Coupled Hartree-Fock-Bogoliubov kinetic equations for a trapped Bose gas

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(February 1, 2008)

Abstract

Using the Kadanoff-Baym non-equilibrium Green's function formalism, we derive the self-consistent Hartree-Fock-Bogoliubov (HFB) collisionless kinetic equations and the associated equation of motion for the condensate wavefunction for a trapped Bose-condensed gas. Our work generalizes earlier work by Kane and Kadanoff (KK) for a uniform Bose gas. We include the off-diagonal (anomalous) pair correlations, and thus we have to introduce an off-diagonal distribution function in addition to the normal (diagonal) distribution function. This results in two coupled kinetic equations. If the off-diagonal distribution function can be neglected as a higher-order contribution, we obtain the semi-classical kinetic equation recently used by Zaremba, Griffin and Nikuni (based on the simpler Popov approximation). We discuss the static local equilibrium solution of our coupled HFB kinetic equations within the semi-classical approximation. We also verify that a solution is the rigid in-phase oscillation of the equilibrium condensate and non-condensate density profiles,

oscillating with the trap frequency.

PACS numbers: 03.75.Fi, 05.20.Dd, 5.30.-b

I. INTRODUCTION

In their classic book, Kadanoff and Baym (KB) [1] developed a systematic way of deriving quantum Boltzmann equations using non-equilibrium Green's functions. KB derived generalized Boltzmann equations for interacting Bose and Fermi gases in the normal phase (no broken symmetry). The advantage of this approach is that one can generate kinetic equations starting from a well-defined single-particle self-energies using functional derivatives. The KB method was generalized by Kane and Kadanoff (KK) [2] to deal with a Bose-condensed gas, with the goal of using the resulting kinetic equations to derive the two-fluid hydrodynamic equations of Landau. The KK analysis worked with diagonal and off-diagonal self-energies, and included terms of second order in the two-particle interaction (the Beliaev second order self-energy diagrams [3]). This gives kinetic equations which include the effect of collisions between the atoms.

In the present paper, we use the KK approach to deal with a trapped Bose-condensed gas. For simplicity, in the present paper we limit ourselves to Hartree-Fock (HF) self-energies and thus the kinetic equations we obtain are in the collisionless approximation [1]. Because we include both normal and anomalous pair correlations, our self-energies give the so-called Hartree-Fock-Bogoliubov (HFB) approximation, as reviewed by Griffin [4]. While the HFB approximation in a uniform gas is known to lead to a single-particle spectrum with an energy gap in the long wavelength limit, it is a "conserving" approximation which will generate two-particle response functions which satisfy conservation laws (see Refs. [1,5] and Section VI for further discussion).

Our main formal result is given in Section III, namely two coupled kinetic equations for the diagonal (f_1) and off-diagonal (f_2) distribution functions for the excited atoms, in addition to an equation of motion for the condensate order parameter. If f_2 can be ignored, we obtain the semi-classical collisionless kinetic equation for f_1 which has been used in recent

discussions of the non-equilibrium properties of the non-condensed atoms [6,7]. In order to gain some insight into our HFB kinetic equations, in Section IV we discuss the static local equilibrium solutions within the semi-classical approximation and verify that these satisfy our equations. In section V, we show that our equations have a solution which corresponds to a a rigid oscillation of the equilibrium density profiles (normal and anomalous), with a frequency equal to the parabolic trap frequency. This generalized Kohn mode is generic but how it arises as a solution of our HFB kinetic equations gives one insight into their structure.

II. DERIVATION OF HFB EQUATIONS

In terms of Bose quantum field operators, the many-body Hamiltonian $(\hat{K} = \hat{H} - \mu_0 \hat{N})$ describing interacting Bosons confined by an external potential $U_{ext}(\mathbf{r})$ is given by:

$$\hat{K} = \int d\mathbf{r} \psi^\dagger(\mathbf{r}) \left[-\frac{1}{2m} \nabla_{\mathbf{r}}^2 + U_{ext}(\mathbf{r}) - \mu_0 \right] \psi(\mathbf{r}) + \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') v(\mathbf{r} - \mathbf{r}') \psi(\mathbf{r}) \psi(\mathbf{r}').$$

$$(1)$$

For a discussion of the properties of a dilute Bose gas at low temperatures the two-body interaction $v(\mathbf{r} - \mathbf{r}')$ can be effectively treated using the s-wave approximation: $v(\mathbf{r}) = g\delta(\mathbf{r})$, where $g = 4\pi a/m$ (we set $\hbar = 1$ throughout this article). We shall be interested in the 2×2 matrix single-particle Green's function defined by [2,8]

$$\hat{g}(1, 1'; U) = -i \begin{pmatrix} \langle T\psi(1)\psi^\dagger(1') \rangle & \langle T\psi(1)\psi(1') \rangle \\ \langle T\psi^\dagger(1)\psi^\dagger(1') \rangle & \langle T\psi^\dagger(1)\psi(1') \rangle \end{pmatrix}. \qquad (2)$$

Here, T represents the time-ordering operation and we use the usual abbreviated notation [1], $1 \equiv (\mathbf{r}t)$ and $1' \equiv (\mathbf{r}'t')$. We separate out the condensate part of the field operator in the usual fashion

$$\psi(\mathbf{r}) = \langle \psi(\mathbf{r}) \rangle_t + \tilde{\psi}(\mathbf{r}), \qquad (3)$$

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where $\langle \tilde{\psi}(\mathbf{r}) \rangle = 0$ and $\langle \psi(\mathbf{r}) \rangle_t = \Phi(\mathbf{r}, t)$ is the macroscopic wavefunction. The non-condensate (or excited atom component) field operators $\tilde{\psi}(\mathbf{r})$ and $\tilde{\psi}^\dagger(\mathbf{r})$ satisfy the usual Bose commutation relations. Using (3), the matrix in (2) splits into two parts

$$\hat{g}(1, 1'; U) = \hat{\tilde{g}}(1, 1'; U) + \hat{h}(1, 1'; U). \tag{4}$$

Here $\hat{\tilde{g}}$ is identical to (2) except that it involves the non-condensate part of the field operators, and

$$\hat{h}(1, 1'; U) \equiv -i \begin{pmatrix} \Phi(1)\Phi^*(1') & \Phi(1)\Phi(1') \\ \Phi^*(1)\Phi^*(1') & \Phi^*(1)\Phi(1') \end{pmatrix}, \tag{5}$$

with $\langle \psi^\dagger(\mathbf{r}) \rangle_t \equiv \Phi^*(\mathbf{r}, t)$. In a Bose-condensed system, the finite value of $\Phi(\mathbf{r}, t)$ leads to finite values of the off-diagonal (or anomalous) propagators $\langle \tilde{\psi}(1)\tilde{\psi}(1') \rangle$ and $\langle \tilde{\psi}^\dagger(1)\tilde{\psi}^\dagger(1') \rangle$. These must be dealt with on a equal basis with the diagonal (or normal) propagators, which is the reason we must work with a 2×2 matrix single-particle Green's function.

A very convenient and elegant way of generating the equations of motion for $\hat{\tilde{g}}$ and $\hat{h}$ is to use functional derivatives with respect to weak external fields. The latter are described by

$$H'(t) = \int d\mathbf{r}[U(\mathbf{r}, t)\tilde{n}(\mathbf{r}, t) + \eta(\mathbf{r}, t)\tilde{\psi}^\dagger(\mathbf{r}, t) + \eta^*(\mathbf{r}, t)\tilde{\psi}(\mathbf{r}, t)], \tag{6}$$

where $U(\mathbf{r}, t)$ is the external generating scalar field, while $\eta(\mathbf{r}, t)$ and $\eta^*(\mathbf{r}, t)$ describe the symmetry-breaking fields involving "particle sources" [9,10]. The higher-order Green's functions can all be expressed as a functional derivatives of single-particle Green's functions with respect to these fields.

Following the Kane-Kadanoff (KK) analysis [1,2], the HFB equations of motion can be conveniently written in the following 2×2 matrix form

$$\int d\bar{1} \left[\hat{g}_0^{-1}(1, \bar{1}) - \hat{\Sigma}^{HF}(1, \bar{1}) \right] \hat{\tilde{g}}(\bar{1}, 1') = \delta(1, 1'), \tag{7}$$

$$\int d\bar{1}\hat{\tilde{g}}(1,\bar{1})\left[\hat{g}_0^{-1}(\bar{1},1') - \hat{\Sigma}^{HF}(\bar{1},1')\right] = \delta(1,1'), \tag{8}$$

$$\int d\bar{1}\left[\hat{g}_0^{-1}(1,\bar{1}) - \hat{S}^{HF}(1,\bar{1})\right]\hat{h}(\bar{1},1') = -i\hat{\eta}^{ext}(1)\langle\hat{\Psi}^\dagger(1')\rangle, \tag{9}$$

$$\int d\bar{1}\hat{h}(1,\bar{1})\left[\hat{g}_0^{-1}(\bar{1},1') - \hat{S}^{HF}(\bar{1},1')\right] = -i\langle\hat{\Psi}(1)\rangle\hat{\eta}^{ext\dagger}(1'). \tag{10}$$

In the above equations, integration over $d\bar{1}$ means integration over the coordinates $(\mathbf{r}_{\bar{1}}, t_{\bar{1}})$, $\delta(1,1') \equiv \delta(\mathbf{r} - \mathbf{r}')\delta(t - t')$, and we have to introduce spinors

$$\langle\hat{\Psi}(1)\rangle \equiv \begin{pmatrix} \Phi(1) \\ \Phi^*(1) \end{pmatrix}, \quad \langle\hat{\Psi}^\dagger(1')\rangle \equiv (\Phi^*(1), \Phi(1)). \tag{11}$$

Here $\hat{\eta}^{ext}$ describes the external particle-source fields defined in (6),

$$\hat{\eta}^{ext}(1) = \begin{pmatrix} \eta(1) \\ \eta^*(1) \end{pmatrix}. \tag{12}$$

The inverse of the 2×2 matrix non-interacting propagator $g_0(1,1')$ is defined by

$$g_0^{-1}(1,1') = \left[i\tau^{(3)}\frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} - U_{ext}(\mathbf{r}_1) - U(1) + \mu_0\right]\delta(1,1'). \tag{13}$$

In the HFB approximation, the 2×2 self-energies in (7)-(10) are given by [2,4,9]

$$\hat{\Sigma}^{HF}(1,1') = g\begin{pmatrix} 2n(1), & m(1) \\ m^*(1), & 2n(1) \end{pmatrix}\delta(1,1'), \quad \hat{S}^{HF}(1,1') = g\begin{pmatrix} 2\tilde{n}(1) + n_c(1), & \tilde{m}(1) \\ \tilde{m}^*(1), & 2\tilde{n}(1) + n_c(1) \end{pmatrix}\delta(1,1'). \tag{14}$$

In the above equations, $n(1)$ is the *total* local density given by

$$n(1) = i\left[\tilde{g}_{11}^<(1,1^+) + h_{11}(1,1)\right] = \langle\tilde{\psi}^\dagger(1)\tilde{\psi}(1)\rangle + |\Phi(1)|^2 \equiv \tilde{n}(1) + n_c(1), \tag{15}$$

where $\tilde{n}(1)$ and $n_c(1)$ are non-condensate and condensate density, respectively. Similarly, $m(1)$ is the "anomalous" local density defined by

$$m(1) = i\left[\tilde{g}_{12}^<(1,1) + h_{12}(1,1)\right] = \langle\tilde{\psi}(1)\tilde{\psi}(1)\rangle + [\Phi(1)]^2 = \tilde{m}(1) + [\Phi(1)]^2$$

$$m^*(1) = i\left[\tilde{g}_{21}^<(1,1) + h_{21}(1,1)\right] = \langle\tilde{\psi}^\dagger(1)\tilde{\psi}^\dagger(1)\rangle + [\Phi^*(1)]^2 = \tilde{m}^*(1) + [\Phi^*(1)]^2. \quad (16)$$

These HFB results were first written down in this formalism by Kane and Kadanoff [2], with the "particle-source" fields η and η^* left implicit. The equation of motion for the order parameter $\Phi(1)$ in this HFB approximation is given by

$$\left[i\frac{\partial}{\partial t_1} + \frac{\nabla_1^2}{2m} + \mu_0 - U_{ext}(\mathbf{r}_1) - U(1) - g\left[2\tilde{n}(1) + n_c(1)\right]\right]\Phi(1) = g\tilde{m}(1)\Phi^*(1) + \eta(1), \quad (17)$$

and its complex conjugate. Equations (7)-(17) are a closed set of equations and define what is called the dynamic HFB approximation.

We now turn to solving the HFB equations of motion for atoms in the presence of a trapping potential. If the external generating fields induce a disturbance with a wavelength much longer than thermal wavelengths and frequencies much smaller than characteristic particle energies then the propagator $g(1,1') = g(\mathbf{r}, t; \mathbf{R}, T)$ can be expected to vary slowly as a function of the center-of-mass coordinates

$$\mathbf{R} = \frac{1}{2}(\mathbf{r}_1 + \mathbf{r}_{1'}), \quad T = \frac{1}{2}(t_1 + t_{1'}), \quad (18)$$

and to be dominated by small values of the relative coordinates

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_{1'}, \quad t = \frac{1}{2}(t_1 - t_{1'}). \quad (19)$$

More precisely, Fourier transforming with respect to $\mathbf{r}$ and t, the function $g(\mathbf{p}, \omega; \mathbf{R}, T)$ describes the density of elementary excitations of momenta $\mathbf{p}$ and energy ω at point $(\mathbf{R}, T)$ [1]. These quasiparticles are assumed to have high momentum and energy (relative to the collective modes we would obtain from the kinetic equations), which means that $g(\mathbf{r}, t; \mathbf{R}, T)$ is mainly weighted at small values of the coordinates $\mathbf{r}$ and t.

If the Bose-condensate order parameter $\langle\psi(1)\rangle$ is written in terms of amplitude and phase variables

$$\langle \psi(1) \rangle = [n_c(1)]^{\frac{1}{2}} e^{i\theta(1)}, \tag{20}$$

we can generalize the usual definitions for superfluid velocity and local chemical potential to non-equilibrium systems by identifying [2]

$$\nabla_1 \theta(1) = m\mathbf{v}_s(1)$$
$$\frac{\partial \theta(1)}{\partial t_1} = -\left[\mu(1) - \mu_0 + \frac{1}{2}mv_s{}^2(1)\right]. \tag{21}$$

The superfluid velocity $\mathbf{v}_s$ enters as the gradient of the phase of the condensate wavefunction and the local chemical potential $\mu(1)$ is connected with the time-derivative of the phase. However, a problem arises in that the phase is a rapidly varying function of $(\mathbf{R}, T)$ which induces strong variations in the off-diagonal elements of $\hat{h}$, and these are coupled to the components of $\tilde{g}$. To remove this strong $(\mathbf{R}, T)$-dependence associated with the phase θ, we apply the well-known [2] gauge transformations on $\hat{h}(1, 1')$ and $\hat{\tilde{g}}(1, 1')$:

$$\hat{h}'(1, 1') = e^{-i\theta(1)\tau^{(3)}} \hat{h}(1, 1') e^{i\theta(1)\tau^{(3)}}$$
$$\hat{\tilde{g}}'(1, 1') = e^{-i\theta(1)\tau^{(3)}} \hat{\tilde{g}}(1, 1') e^{i\theta(1)\tau^{(3)}}, \tag{22}$$

where $\tau^{(3)}$ is the Pauli spin matrix. The physical interpretation of (22) is that it involves a transformation to a coordinate system in which non-condensate atoms are moving with average velocity $\mathbf{v}_s$ with respect to a stationary condensate. The gauge transformation (22) removes the strong $(\mathbf{R}, T)$-dependence associated with the order parameter phase θ and leaves the equations of motion (7)-(10) invariant if we replace g_0^{-1} in (13) by

$$g'_0{}^{-1}(1, 1') = \left[i\tau^{(3)}\frac{\partial}{\partial t_1} - \frac{\partial \theta(1)}{\partial t_1} + \frac{1}{2}[\nabla_1 + i\nabla_1\theta(1)\tau^{(3)}]^2 - U_{ext}(\mathbf{r_1}) - U(1) + \mu_0\right]\delta(1, 1'). \tag{23}$$

After carrying out this gauge transformation, the HFB equations for $\tilde{g}_{11}^{\gtrless}(1, 1'; U)$ and $\tilde{g}_{12}^{\gtrless}(1, 1'; U)$ given by (7)-(10) become

$$\left[i\frac{\partial}{\partial t_1} - \frac{\partial\theta(1)}{\partial t_1} + \frac{1}{2m}[\nabla_1 + i\nabla_1\theta(1)]^2 + \mu_0 - U_{eff}(1) \right] \tilde{g}'^{\lessgtr}_{11}(1,1';U) = gm'(1)\tilde{g}'^{\lessgtr}_{21}(1,1';U)$$

$$\left[i\frac{\partial}{\partial t_1} - \frac{\partial\theta(1)}{\partial t_1} + \frac{1}{2m}[\nabla_1 + i\nabla_1\theta(1)]^2 + \mu_0 - U_{eff}(1) \right] \tilde{g}'^{\lessgtr}_{12}(1,1';U) = gm'(1)\tilde{g}'^{\lessgtr}_{22}(1,1';U)$$

$$\left[-i\frac{\partial}{\partial t_{1'}} - \frac{\partial\theta(1')}{\partial t_{1'}} + \frac{1}{2m}[\nabla_{1'} - i\nabla_{1'}\theta(1')]^2 + \mu_0 - U_{eff}(1') \right] \tilde{g}'^{\lessgtr}_{11}(1,1';U) = gm'^*(1')\tilde{g}'^{\lessgtr}_{12}(1,1';U)$$

$$\left[i\frac{\partial}{\partial t_{1'}} - \frac{\partial\theta(1')}{\partial t_{1'}} + \frac{1}{2m}[\nabla_{1'} + i\nabla_{1'}\theta(1')]^2 + \mu_0 - U_{eff}(1') \right] \tilde{g}'^{\lessgtr}_{12}(1,1';U) = gm'(1')\tilde{g}'^{\lessgtr}_{11}(1,1';U). \quad (24)$$

Here

$$U_{eff}(1) \equiv U_{ext}(\mathbf{r}_1) + U(1) + 2gn'(1) \tag{25}$$

is the effective self-consistent Hartree-Fock dynamic mean field. The condensate part $2gn_c(1)$ in (25) can be viewed as an additional "external field" acting on the non-condensate. Since we will work with these gauge-transformed correlation functions in the rest of this paper, we drop the primes on $\tilde{g}_{11}$, $\tilde{g}_{12}$, n and m to simplify the notation.

The corresponding equation of motion for the condensate amplitude $\sqrt{n_c(1)}$ is in the moving frame of reference can be written in the form:

$$\left[i\frac{\partial}{\partial t_1} - \frac{\partial\theta(1)}{\partial t_1} + \frac{\nabla_1^2}{2m} - \frac{1}{2}m\mathbf{v}_s^2 + \mu_0 - U_{ext}(\mathbf{r}_1) - U(1) - g\left[2\tilde{n}(1) + n_c(1)\right] + \right.$$
$$\left. + i\mathbf{v}_s(1)\cdot\nabla_1 + \frac{i}{2}\nabla_1\cdot\mathbf{v}_s(1) \right] \sqrt{n_c(1)} = g\tilde{m}(1)\sqrt{n_c(1)} + \eta'(1), \tag{26}$$

where $\eta'(1) \equiv \eta(1)e^{-i\theta(1)}$ is the symmetry-breaking source function in the moving frame of reference. We note that there is a factor of two difference between how the condensate and non-condensate enter these equations. This is because atoms in the condensate are in the same state and thus there is no exchange part. In the case of non-condensate atoms, both Hartree and Fock terms arise since we are dealing with atoms in different states.

III. HFB KINETIC EQUATIONS

To rewrite the equations of motion derived in Section II in the form of kinetic equations, we recall the connection between the usual single-particle distribution function $f_1(\mathbf{p}, \mathbf{R}, T)$ and the diagonal Green's function $\tilde{g}_{\alpha\alpha}^<(1, 1')$. We define (see p. 67 of Ref. [1]):

$$\begin{aligned} f_1(\mathbf{p}, \mathbf{R}, T) &\equiv \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \left[i\tilde{g}_{11}^<(\mathbf{r}, t = 0; \mathbf{R}, T) \right] \\ &= \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \langle \tilde{\psi}_U^\dagger(\mathbf{R} - \frac{\mathbf{r}}{2}, T) \tilde{\psi}_U(\mathbf{R} + \frac{\mathbf{r}}{2}, T) \rangle \end{aligned} \qquad (27)$$

where, by definition,

$$\int \frac{d\mathbf{p}}{(2\pi)^3} f_1(\mathbf{p}, \mathbf{R}, T) = \tilde{n}(\mathbf{R}, T). \qquad (28)$$

We can see that $f_1(\mathbf{p}, \mathbf{R}, T)$ corresponds to the well-known Wigner distribution function. In the classical limit, it reduces to the distribution function giving the number of atoms with momentum $\mathbf{p}$ at point $\mathbf{R}$ and time T. The symmetry-breaking terms in (6) result in the finite value of anomalous Green's function, and thus it is natural to introduce an additional distribution function for the non-condensate atoms which will give us the anomalous non-condensate density $\tilde{m}(1)$, namely

$$\begin{aligned} f_2(\mathbf{p}, \mathbf{R}, T) &\equiv \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \left[i\tilde{g}_{12}^<(\mathbf{r}, t = 0; \mathbf{R}, T) \right] \\ &= \int d\mathbf{r} e^{-i\mathbf{p}\cdot\mathbf{r}} \langle \tilde{\psi}_U(\mathbf{R} - \frac{\mathbf{r}}{2}, T) \tilde{\psi}_U(\mathbf{R} + \frac{\mathbf{r}}{2}, T) \rangle. \end{aligned} \qquad (29)$$

One can easily verify that the pair correlation function $\tilde{m}$ in (16) is given by

$$\int \frac{d\mathbf{p}}{(2\pi)^3} f_2(\mathbf{p}, \mathbf{R}, T) = \tilde{m}(\mathbf{R}, T). \qquad (30)$$

It is important to remember that, as defined, the distribution functions f_1 and f_2 describe the behaviour of the atoms. They should not be confused with the distribution function for the quasiparticle excitations, such as discussed in Ref. [11].

To obtain kinetic equations, we follow the Kadanoff-Baym approach in [1,2] and rewrite the equations for $\tilde{g}_{11}^{<}(1,1')$ and $\tilde{g}_{12}^{<}(1,1')$ in the relative and center-of-mass coordinates $(\mathbf{r}, t; \mathbf{R}, T)$. We could obtain equations for $g(\mathbf{r}, t; \mathbf{R}, T)$ as in Ref. [2], but for the simple HFB self-energies of interest it is sufficient to consider $t=0$, $i.e.$ set $t_{1'} = t_1^+ = T$ (see Ch. 7 of [1]). Using (24), this procedure gives

$$
\begin{aligned}
&\left[i\frac{\partial}{\partial T} + \Delta U_{eff}^{\mu}(\mathbf{r}, \mathbf{R}, T) + \frac{1}{m}\nabla_{\mathbf{r}} \cdot \nabla_{\mathbf{R}} + i\mathbf{v}_s^{-}(\mathbf{r}, \mathbf{R}, T) \cdot \nabla_{\mathbf{r}} + \frac{i}{2}\mathbf{v}_s^{+}(\mathbf{r}, \mathbf{R}, T) \cdot \nabla_{\mathbf{R}} \right. \\
&\left. + \frac{i}{2}\nabla_{\mathbf{r}} \cdot \mathbf{v}_s^{-}(\mathbf{r}, \mathbf{R}, T) + \frac{i}{4}\nabla_{\mathbf{R}} \cdot \mathbf{v}_s^{+}(\mathbf{r}, \mathbf{R}, T) \right] \tilde{g}_{11}^{<}(\mathbf{r}, \mathbf{R}, T) \\
&= g\left[m(\mathbf{R}+\frac{\mathbf{r}}{2}, T)\tilde{g}_{21}^{<}(\mathbf{r}, \mathbf{R}, T) - m^*(\mathbf{R}-\frac{\mathbf{r}}{2}, T)\tilde{g}_{12}^{<}(\mathbf{r}, \mathbf{R}, T) \right],
\end{aligned}
\tag{31}
$$

where we have introduced the abbreviations

$$
\Delta U_{eff}^{\mu}(1,1') \equiv \mu(1) - \mu(1') - U_{eff}(1) + U_{eff}(1'),
\tag{32}
$$

and

$$
\mathbf{v}_s^{\pm}(\mathbf{r}, \mathbf{R}, T) \equiv \mathbf{v}_s(\mathbf{R}+\frac{\mathbf{r}}{2}, T) \pm \mathbf{v}_s(\mathbf{R}-\frac{\mathbf{r}}{2}, T).
\tag{33}
$$

Similarly, the equation of motion for $\tilde{g}_{12}^{<}(1,1')$ expressed in the $(\mathbf{r}, \mathbf{R}; t, T)$ variables gives

$$
\begin{aligned}
&\left[i\frac{\partial}{\partial T} + \mu(\mathbf{R}+\frac{\mathbf{r}}{2}, T) + \mu(\mathbf{R}-\frac{\mathbf{r}}{2}, T) - U_{eff}(\mathbf{R}+\frac{\mathbf{r}}{2}, T) - U_{eff}(\mathbf{R}-\frac{\mathbf{r}}{2}, T) + \frac{\nabla_{\mathbf{R}}^2}{4m} + \right. \\
&+ \frac{1}{m}\nabla_{\mathbf{r}}^2 + i\mathbf{v}_s^{-}(\mathbf{r}, \mathbf{R}, T) \cdot \nabla_{\mathbf{r}} + \frac{i}{2}\mathbf{v}_s^{+}(\mathbf{r}, \mathbf{R}, T) \cdot \nabla_{\mathbf{R}} + \frac{i}{2}\nabla_{\mathbf{r}} \cdot \mathbf{v}_s^{-}(\mathbf{r}, \mathbf{R}, T) + \\
&\left. + \frac{i}{4}\nabla_{\mathbf{R}} \cdot \mathbf{v}_s^{+}(\mathbf{r}, \mathbf{R}, T) \right] \tilde{g}_{12}^{<}(\mathbf{r}, \mathbf{R}, T) = g\left[m(\mathbf{R}+\frac{\mathbf{r}}{2}, T)\tilde{g}_{22}^{<}(\mathbf{r}, \mathbf{R}, T) + m(\mathbf{R}-\frac{\mathbf{r}}{2}, T)\tilde{g}_{11}^{<}(\mathbf{r}, \mathbf{R}, T) \right].
\end{aligned}
\tag{34}
$$

Finally, the equation of motion (26) for the *amplitude* of the order parameter can be written in the $(\mathbf{R}, T)$ coordinates as

$$
\begin{aligned}
&\left[i\frac{\partial}{\partial T} - \frac{\partial\theta(\mathbf{R}, T)}{\partial T} + \frac{\nabla_{\mathbf{R}}^2}{2m} - \frac{1}{2}m\mathbf{v}_s^2(\mathbf{R}, T) + \mu_0 - U_{ext}(\mathbf{R}) - U(\mathbf{R}, T) \right. \\
&\left. - g\left[2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T)\right] + i\mathbf{v}_s(\mathbf{R}, T) \cdot \nabla_{\mathbf{R}} + \frac{i}{2}\nabla_{\mathbf{R}} \cdot \mathbf{v}_s(\mathbf{R}, T) \right] \sqrt{n_c(\mathbf{R}, T)} \\
&= g\tilde{m}(\mathbf{R}, T)\sqrt{n_c(\mathbf{R}, T)} + \eta'(\mathbf{R}, T).
\end{aligned}
\tag{35}
$$

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The coupled set of HFB equations given by (31),(34) and (35) are the main formal results of this paper. These are a straightforward generalization of the analogous equations in the normal Hartree-Fock approximation discussed by Kadanoff and Baym (see Eq. (7.7) on pg. 71 of Ref. [1]). These results are important since they allow us to go beyond the simple HFP approximation which has been the basis of recent work on the non-equilibrium properties of a trapped Bose-condensed gas (see, for example, [6,7]).

We next proceed to use (31) and (34) to derive self-consistent equations for $f_1(\mathbf{p}, \mathbf{R}, T)$ and $f_2(\mathbf{p}, \mathbf{R}, T)$ for the case when the external perturbation varies slowly in space and time. In this case, we expect that physical quantities $\mathbf{v}_s$, μ, U, U_{ext}, $n(1)$, and $m(1)$ all vary slowly as functions of the center-of-mass coordinates $(\mathbf{R}, T)$. Thus, in the lowest approximation, and using the fact that small values of $\mathbf{r}$ are most important, we can use

$$\mathbf{v}_s\left(\mathbf{R} \pm \frac{\mathbf{r}}{2}, T\right) = \mathbf{v}_s(\mathbf{R}, T) \pm \left[\frac{\mathbf{r}}{2} \cdot \nabla_\mathbf{R}\right] \mathbf{v}_s(\mathbf{R}, T), \tag{36}$$

and hence

$$\mathbf{v}_s^+(\mathbf{r}, \mathbf{R}, T) \simeq 2\mathbf{v}_s(\mathbf{R}, T), \qquad \mathbf{v}_s^-(\mathbf{r}, \mathbf{R}, T) \simeq [\mathbf{r} \cdot \nabla_\mathbf{R}] \mathbf{v}_s(\mathbf{R}, T),$$

$$\Delta U_{eff}^\mu(\mathbf{r}, \mathbf{R}, T) \simeq -\mathbf{r} \cdot \nabla_\mathbf{R}[U_{eff}(\mathbf{R}, T) - \mu(\mathbf{R}, T)]. \tag{37}$$

If we rewrite (31) using (27) and (29), and Fourier transform it, we obtain after some algebra

$$\left[\frac{\partial}{\partial T} - \nabla_\mathbf{R}[\tilde{\epsilon}_p + \mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_\mathbf{p} + \nabla_\mathbf{p}[\tilde{\epsilon}_p + \mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_\mathbf{R}\right] f_1(\mathbf{p}, \mathbf{R}, T)$$

$$= -ig \left[m(\mathbf{R}, T)f_2(-\mathbf{p}, \mathbf{R}, T) - m^*(\mathbf{R}, T)f_2(\mathbf{p}, \mathbf{R}, T)\right]$$

$$+ \frac{g}{2} \left[\nabla_\mathbf{R} m(\mathbf{R}, T) \cdot \nabla_\mathbf{p} f_2(-\mathbf{p}, \mathbf{R}, T) + \nabla_\mathbf{R} m^*(\mathbf{R}, T) \cdot \nabla_\mathbf{p} f_2(\mathbf{p}, \mathbf{R}, T)\right], \tag{38}$$

where the "normal" single-particle energy is defined by

$$\tilde{\epsilon}_p(\mathbf{R}, T) \equiv \frac{p^2}{2m} + U_{eff}(\mathbf{R}, T) - \mu(\mathbf{R}, T). \tag{39}$$

We emphasize that $\tilde{\epsilon}_p$ is not, in general, the local HFB excitation energy. The corresponding kinetic equation for $f_2(\mathbf{p}, \mathbf{R}, T)$ is:

$$\left[\frac{\partial}{\partial T} + i2\tilde{\epsilon}_p(\mathbf{R}, T) - \nabla_{\mathbf{R}} \left[\mathbf{v}_s \cdot \mathbf{p} \right] \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}} \left[\mathbf{v}_s \cdot \mathbf{p} \right] \cdot \nabla_{\mathbf{R}} \right] f_2(\mathbf{p}, \mathbf{R}, T) =$$

$$-igm(\mathbf{R}, T) \left[f_1(\mathbf{p}, \mathbf{R}, T) + f_1(-\mathbf{p}, \mathbf{R}, T) + 1 \right]$$

$$+ \frac{g}{2} \nabla_{\mathbf{R}} m(\mathbf{R}, T) \cdot \nabla_{\mathbf{p}} \left[f_1(\mathbf{p}, \mathbf{R}, T) - f_1(-\mathbf{p}, \mathbf{R}, T) \right]. \tag{40}$$

These are the coupled HFB collisionless kinetic equations in a frame moving with the velocity $\mathbf{v}_s$, for the case of slowly varying disturbances.

The equation of motion (35) for the amplitude of the order parameter is an exact equation, and is not limited for a slowly varying disturbances. Equating the real and imaginary parts of (35), we obtain two hydrodynamic equations of motion for the condensate

$$\frac{\partial n_c(\mathbf{R}, T)}{\partial T} = -\nabla_{\mathbf{R}} \left[n_c(\mathbf{R}, T) \mathbf{v}_s(\mathbf{R}, T) \right] + 2g n_c(\mathbf{R}, T) Im \left[\tilde{m}(\mathbf{R}, T) \right], \tag{41}$$

$$\frac{\partial \theta(\mathbf{R}, T)}{\partial T} + \frac{1}{2} m \mathbf{v}_s^2(\mathbf{R}, T) - \mu_0 = -\mu(\mathbf{R}, T) - \eta'(\mathbf{R}, T) \frac{1}{\sqrt{n_c(\mathbf{R}, T)}}, \tag{42}$$

where the condensate chemical potential $\mu(\mathbf{R}, T)$ is defined by

$$\mu(\mathbf{R}, T) \equiv -\frac{\nabla_{\mathbf{R}}^2 \sqrt{n_c(\mathbf{r}, T)}}{2m \sqrt{n_c(\mathbf{R}, T)}} + U_{ext}(\mathbf{R}) + U(\mathbf{R}, T) + g \left[2\tilde{n}(\mathbf{R}, T) + n_c(\mathbf{R}, T) \right] + g Re \left[\tilde{m}(\mathbf{R}, T) \right]. \tag{43}$$

Taking the gradient of (42), we obtain the generalized Landau equation [6]

$$m \left[\frac{\partial \mathbf{v}_s(\mathbf{R}, T)}{\partial T} + \frac{1}{2} \nabla_{\mathbf{R}} \mathbf{v}_s^2(\mathbf{R}, T) \right] = -\nabla_{\mathbf{R}} \left[\mu(\mathbf{R}, T) + \eta'(\mathbf{R}, T) \frac{1}{\sqrt{n_c(\mathbf{R}, T)}} \right]. \tag{44}$$

In ZGN [6], the contribution of the external symmetry-breaking field η' was left implicit. In the often used Thomas-Fermi (TF) approximation, the kinetic energy of the condensate is omitted and the HFB approximation for the condensate chemical potential then simplifies to

$$\mu^{TF}(\mathbf{R}, T) = U_{ext}(\mathbf{R}) + 2g\tilde{n}(\mathbf{R}, T) + g n_c(\mathbf{R}, T) + g Re \left[\tilde{m}(\mathbf{R}, T) \right]. \tag{45}$$

Next we consider Eq.(38) in the high temperature limit. In that case, n_c is small and therefore we can neglect the terms proportional to $gn_c f_2$ in (38) as small. Also, we know that $\tilde{m}$ must be at least of order g [12], and therefore the $g\tilde{m}$ contribution to the self-energy is $\mathcal{O}(g^2)$. Thus, we can neglect the right side of (38), leaving

$$\left[\frac{\partial}{\partial T} - \nabla_\mathbf{R}[\tilde{\epsilon}_p + \mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_\mathbf{p} + \nabla_\mathbf{p}[\tilde{\epsilon}_p + \mathbf{v}_s \cdot \mathbf{p}] \cdot \nabla_\mathbf{R} \right] f_1(\mathbf{p}, \mathbf{R}, T) = 0. \qquad (46)$$

This is precisely the expected collisionless Boltzmann equation for f_1, valid in the HFP approximation. This approximation is only valid at finite temperatures, close to T_{BEC}, in which case $\tilde{\epsilon}_p(\mathbf{R}, T)$ as defined in (39) *is* the correct excitation energy. In this limit, the kinetic equation (46) becomes equivalent to that derived (using a different formalism) by Kirkpatrick and Dorfman (KD) [11] for a uniform gas. It is the local rest frame equivalent of the kinetic equation used by ZGN [6] if one ignores the collision terms.

IV. STATIC HFB EQUILIBRIUM SOLUTIONS IN THE SEMI-CLASSICAL APPROXIMATION

For a uniform system, the matrix Green's functions defined in (4) depend only on the relative coordinates, i.e. $\tilde{g}_{\alpha\beta}(1, 1') = \tilde{g}_{\alpha\beta}(1 - 1')$. One can then solve (7)-(10) by Fourier transformation to obtain expressions for $\tilde{g}_{11}(\mathbf{p}, \omega)$ and $\tilde{g}_{12}(\mathbf{p}, \omega)$. Using these, we obtain the single-particle spectral density in the form [8]

$$\begin{aligned}
A_{11}(\mathbf{p}, \omega) &= -2Im\tilde{g}_{11}(\mathbf{p}, \omega + i0^+) = -2Im\left[\frac{u_p^2}{\omega - E_p + i0^+} - \frac{v_p^2}{\omega + E_p + i0^+} \right] \\
&= 2\pi \left[u_p^2 \delta(\omega - E_p) - v_p^2 \delta(\omega + E_p) \right] \\
A_{12}(\mathbf{p}, \omega) &= -2Im\tilde{g}_{12}(\mathbf{p}, \omega + i0^+) = 2Im\left[\frac{u_p v_p^*}{\omega - E_p + i0^+} - \frac{u_p v_p^*}{\omega + E_p + i0^+} \right] \\
&= -2\pi u_p v_p^* \left[\delta(\omega - E_p) - \delta(\omega + E_p) \right].
\end{aligned} \qquad (47)$$

The HFB excitation energy E_p is given by

$$E_p^2 = \left[\frac{p^2}{2m} + 2gn - \mu\right]^2 - (gm)^2, \tag{48}$$

with

$$\tilde{\epsilon}_p \equiv \frac{p^2}{2m} + 2gn - \mu, \tag{49}$$

and

$$u_p^2 = \frac{1}{2}\left[\frac{\tilde{\epsilon}_p}{E_p} + 1\right], \quad v_p^2 = \frac{1}{2}\left[\frac{\tilde{\epsilon}_p}{E_p} - 1\right], \quad u_p v_p^* = \frac{gm}{2E_p}. \tag{50}$$

The results in (47) have the same structure as in the simpler Bogoliubov approximation (see Ch.14 of Ref [8]). From (17), with $\Phi(1) = const.$, it follows that chemical potential in HFB approximation is given by [4,9]

$$\mu = g\left(n + \tilde{n} + \tilde{m}\right). \tag{51}$$

If we use this result in (48), it reduces to

$$E_{p=0}^2 = g^2\left[n_c - \tilde{m}\right]^2 - g^2\left[n_c + \tilde{m}\right]^2 = -4g^2\tilde{m}n_c. \tag{52}$$

Therefore, the long-wavelength HFB excitation spectrum has a finite energy gap.

The density of non-condensate atoms in a uniform system can be found using the single-particle spectral density given in (47)

$$\begin{aligned}
\tilde{n} &= \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\omega}{2\pi} N_0(\omega) A_{11}(\mathbf{p}, \omega) \\
&= \int \frac{d\mathbf{p}}{(2\pi)^3} \left[v_p^2 + \left(u_p^2 + v_p^2\right) N_0(E_p)\right] \\
&= \int \frac{d\mathbf{p}}{(2\pi)^3} \left[\frac{\tilde{\varepsilon}_p}{2E_p}\left[2N_0(E_p) + 1\right] - \frac{1}{2}\right],
\end{aligned} \tag{53}$$

where $N_0(E_p)$ is the Bose distribution function. Similarly, the anomalous density is given by

$$\tilde{m} = \int \frac{d\mathbf{p}}{(2\pi)^3} \int \frac{d\omega}{2\pi} N_0(\omega) A_{12}(\mathbf{p}, \omega)$$

$$= -\int \frac{d\mathbf{p}}{(2\pi)^3} v_p u_p^* \left[2N_0(E_p) + 1 \right]$$

$$= -\int \frac{d\mathbf{p}}{(2\pi)^3} \frac{gm}{2E_p} \left[2N_0(E_p) + 1 \right]. \tag{54}$$

We want to find an approximate solution for f_1 and f_2 for a trapped Bose gas which will be the analogue of the uniform gas results in (53) and (54). We start from the coupled static HFB equations, as derived in Ref. [4]:

$$\hat{\mathcal{L}} u_i(\mathbf{R}) - gm_0(\mathbf{R}) v_i(\mathbf{R}) = E_i u_i(\mathbf{R})$$

$$\hat{\mathcal{L}} v_i(\mathbf{R}) - gm_0(\mathbf{R}) u_i(\mathbf{R}) = -E_i v_i(\mathbf{R}), \tag{55}$$

where

$$\hat{\mathcal{L}} = -\frac{\nabla^2}{2m} + U_{ext}(\mathbf{R}) - \mu + 2gn(\mathbf{R}). \tag{56}$$

We can solve (55) using the well-known semi-classical approximation. We assume that the normal and anomalous density are smooth functions of $\mathbf{R}$ on the scale of length of $a_{HO} \equiv \sqrt{\frac{\hbar}{m\omega_0}}$, which defines the size of the condensate in a harmonic potential with a trap frequency ω_0 (we reinsert $\hbar$ in this discussion for physical clarity). Hence the self-consistent Hartree-Fock mean field varies slowly (as a function of $\mathbf{R}$) on the length scale of order a_{HO}. Therefore, we can assume that $u_i(\mathbf{R})$ and $v_i(\mathbf{R})$ have a form of a plane waves with a slowly varying amplitude in that region [13], i.e.

$$u_i(\mathbf{R}) \equiv u(\mathbf{p}, \mathbf{R}) e^{i\varphi(\mathbf{R})}$$

$$v_i(\mathbf{R}) \equiv v(\mathbf{p}, \mathbf{R}) e^{i\varphi(\mathbf{R})}. \tag{57}$$

We introduce the momentum of elementary excitations by

$$\mathbf{p} = \hbar \nabla_{\mathbf{R}} \varphi(\mathbf{R}), \tag{58}$$

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which satisfy the condition $\mathbf{p} \gg \hbar/a_{HO}$. This condition, when expressed in terms of wavelength reduces to the small wavelength limit ($\lambda \ll a_{H0}$), or equivalently, to the semi-classical approximation limit expressed as $kT \gg \hbar\omega_0$, where $\hbar\omega_0$ gives the harmonic well energy level spacing. If this condition is satisfied, we can neglect the spatial derivatives of u and v, as well as the second spatial derivative of φ. This is consistent with a so-called *quasi-classical condition*, which requires that a spatial change in wavelength of the particle must satisfy the condition $d\lambda/dx \ll 1$ [14]. The assumed form given by (57) is only valid in the regions of space where this condition is satisfied. To treat the condensate in the corresponding approximation, we use the Thomas-Fermi approximation which is valid in the large N limit. The only region where the TF approximation for the order parameter is inadequate is close to the classical turning points at the condensate boundary [13,15], which is consistent with inapplicability of the semi-classical approximation near these points.

Putting all this together, we can easily solve the coupled equations (55) for $u(\mathbf{p}, \mathbf{R})$ and $v(\mathbf{p}, \mathbf{R})$,

$$u^2(\mathbf{p}, \mathbf{R}) = \frac{\tilde{\epsilon}_p(\mathbf{R}) + E_p(\mathbf{R})}{2E_p(\mathbf{R})}, \quad v^2(\mathbf{p}, \mathbf{R}) = \frac{\tilde{\epsilon}_p(\mathbf{R}) - E_p(\mathbf{R})}{2E_p(\mathbf{R})}, \quad u(\mathbf{p}, \mathbf{R})v^*(\mathbf{p}, \mathbf{R}) = \frac{gm_0\mathbf{R})}{2E_p(\mathbf{R})}.$$

(59)

The local HFB quasiparticle energy $E_p(\mathbf{R})$ is given by

$$E_p^2(\mathbf{R}) = \tilde{\epsilon}_p^2(\mathbf{R}) - (gm_0(\mathbf{R}))^2 = \left[\frac{p^2}{2m} - \mu_0 + U_{ext}(\mathbf{R}) + 2gn_0(\mathbf{R})\right]^2 - (gm_0(\mathbf{R}))^2. \quad (60)$$

We use the expressions for $\tilde{n}$ and $\tilde{m}$ given in terms of u and v [4]

$$\tilde{n}(\mathbf{R}) = \sum_i \left(\left[\mid u_i(\mathbf{R}) \mid^2 + \mid v_i(\mathbf{R}) \mid^2 \right] N_0(E_i) + \mid v_i(\mathbf{R}) \mid^2 \right),$$

$$\tilde{m}(\mathbf{R}) = -\sum_i u_i(\mathbf{R})v_i^*(\mathbf{R}) \left[2N_0(E_i) + 1 \right], \quad (61)$$

where $N_0(E)$ is the Bose distribution for the quasiparticle excitations. The sum over the quantum states is replaced by the integral $\int d\mathbf{p}/(2\pi)^3$ and the semi-classical approximation for the diagonal distribution function $f_{10}(\mathbf{p}, \mathbf{R})$ is given by

$$f_{10}(\mathbf{p}, \mathbf{R}) = v^2(\mathbf{p}, \mathbf{R}) + \left[u^2(\mathbf{p}, \mathbf{R}) + v^2(\mathbf{p}, \mathbf{R})\right] N_0(E_p(\mathbf{R}))$$

$$= \frac{\tilde{\epsilon}_p(\mathbf{R})}{2E_p(\mathbf{R})} \left[2N_0(E_p(\mathbf{R})) + 1\right] - \frac{1}{2}, \tag{62}$$

while for the off-diagonal distribution function $f_{20}(\mathbf{p}, \mathbf{R})$ we obtain

$$f_{20}(\mathbf{p}, \mathbf{R}) = -\frac{gm_0(\mathbf{R})}{2E_p(\mathbf{R})} \left[2N_0(E_p(\mathbf{R})) + 1\right]. \tag{63}$$

In summary, in the semi-classical approximation, the local static equilibrium normal density is given by

$$\tilde{n}_0(\mathbf{R}) = \int \frac{d\mathbf{p}}{(2\pi)^3} \left[\frac{\tilde{\epsilon}_p(\mathbf{R})}{2E_p(\mathbf{R})} \left[2N_0(E_p(\mathbf{R}) + 1\right] - \frac{1}{2}\right], \tag{64}$$

while the local static equilibrium anomalous density by

$$\tilde{m}_0(\mathbf{R}) = -gm_0(\mathbf{R}) \int \frac{d\mathbf{p}}{(2\pi)^3} \left[\frac{2N_0(E_p(\mathbf{R})) + 1}{2E_p(\mathbf{R})}\right]. \tag{65}$$

As expected, these semi-classical approximation results are the natural generalizations of the results obtained for a *uniform* gas given by (53) and (54). The same kind of semi-classical results have also been obtained in Ref. [13] for a trapped Bose gas using the Popov approximation (which corresponds to the HFB with $\tilde{m}=0$). Since the local quasiparticle energy $E_p(\mathbf{R})$ given by (60) depends on the normal and anomalous densities, the quantities in (60), (64) and (65) must be solved self-consistently, as in Ref. [13].

We can now show that these semi-classical static HFB results for f_{10} and f_{20} satisfy our collisionless static HFB kinetic equations. First of all, we note that in the static limit, (40) reduces to

$$2\tilde{\epsilon}_p(\mathbf{R})f_{20}(\mathbf{p}, \mathbf{R}) = -gm_0(\mathbf{R}) \left[f_{10}(\mathbf{p}, \mathbf{R}) + f_{10}(-\mathbf{p}, \mathbf{R}) + 1\right]$$

$$-i\frac{g}{2}\nabla_{\mathbf{R}}m(\mathbf{R}, T) \cdot \nabla_{\mathbf{p}} \left[f_{10}(\mathbf{p}, \mathbf{R}) - f_{10}(-\mathbf{p}, \mathbf{R})\right]. \tag{66}$$

In static equilibrium without any mass current, we have $f_{10}(\mathbf{p}, \mathbf{R}) = f_{10}(-\mathbf{p}, \mathbf{R})$, and thus (66) can be further simplified to

$$f_{20}(\mathbf{p}, \mathbf{R}) = -\frac{gm_0(\mathbf{R})}{2\tilde{\epsilon}_p(\mathbf{R})} \left[2f_{10}(\mathbf{p}, \mathbf{R}) + 1 \right]. \tag{67}$$

We note that the HFB approximation for $\tilde{\epsilon}_p(\mathbf{R})$ in (39) is given by

$$\tilde{\epsilon}_p(\mathbf{R}) = \frac{p^2}{2m} + \frac{\nabla_{\mathbf{R}}^2 \sqrt{n_{c0}(\mathbf{R})}}{2m\sqrt{n_{c0}(\mathbf{R})}} + g\left[n_{c0}(\mathbf{R}) - \tilde{m}_0(\mathbf{R}) \right]. \tag{68}$$

Similarly, in static thermal equilibrium, Eq.(38) for the diagonal distribution function $f_1(\mathbf{p}, \mathbf{R}, T)$ reduces to

$$\left[-\nabla_{\mathbf{R}} \tilde{\epsilon}_p(\mathbf{R}) \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}} \tilde{\epsilon}_p(\mathbf{R}) \cdot \nabla_{\mathbf{R}} \right] f_{10}(\mathbf{p}, \mathbf{R}) = g\nabla_{\mathbf{R}} m_0(\mathbf{R}) \cdot \nabla_{\mathbf{p}} f_{20}(\mathbf{p}, \mathbf{R}). \tag{69}$$

In the last step, we again have used that $f_{20}(-\mathbf{p}, \mathbf{R}) = f_{20}(\mathbf{p}, \mathbf{R})$ and that m_0 is real.

The results discussed above for $f_{10}(\mathbf{p}, \mathbf{R})$ and $f_{20}(\mathbf{p}, \mathbf{R})$ describe local thermal equilibrium distributions functions induced by collisions, which are not included in our HFB collisionless equations. However, these local equilibrium functions do satisfy the static HFB kinetic equations. Inserting (62) into (66), we obtain (63). This shows that the static equilibrium kinetic equation (67) for the off-diagonal distribution function has a solution consistent with the semi-classical approximation. Substituting the local equilibrium distribution functions f_{10} and f_{20} given by (62) and (63) , a lengthy but straightforward calculation shows that they satisfy the static HFB kinetic equation (69).

We recall that the semi-classical local equilibrium expressions for f_1 and f_2 are only valid under the condition $k_B T \gg \hbar\omega_0$ and $E_{\mathbf{p}}(\mathbf{R}) \gg \hbar\omega_0$. This means that these approximate forms cease to be valid at very low temperatures (for further discussion, see Ref. [13]).

V. KOHN MODE

In this section, following the approach used in Ref. [16], we prove that our coupled equations exhibit the rigid in-phase oscillation (or Kohn mode) [6]. This rigid in-phase center-of-mass oscillation of the condensate and the non-condensate corresponds to

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$$n_c(\mathbf{R}, T) \equiv n_{c0}(\mathbf{R} - \boldsymbol{\eta}(T))$$

$$\tilde{n}(\mathbf{R}, T) \equiv \tilde{n}_0(\mathbf{R} - \boldsymbol{\eta}(T))$$

$$\tilde{m}(\mathbf{R}, T) \equiv \tilde{m}_0(\mathbf{R} - \boldsymbol{\eta}(T)). \tag{70}$$

Here the center-of-mass displacement $\boldsymbol{\eta}(T)$ (with $\mathbf{v}_s = \mathbf{v}_n = \dot{\boldsymbol{\eta}}$) is independent of position and satisfies the harmonic oscillator equation of motion

$$m\frac{\partial^2 \eta_\alpha}{\partial T^2} = -\omega_\alpha^2 \eta_\alpha, \tag{71}$$

where ω_α is the harmonic well trap frequency in the αth direction.

We recall equation of motion for the superfluid velocity given by (44). For the case of a rigid in-phase oscillation described by (70), this equation can be written in the form

$$m\frac{\partial^2 \boldsymbol{\eta}}{\partial T^2} = -\nabla_{\mathbf{R}} \left[\mu_0(\mathbf{R} - \boldsymbol{\eta}(T)) + U_{ext}(\mathbf{R}) - U_{ext}(\mathbf{R} - \boldsymbol{\eta}(T)) \right]. \tag{72}$$

Since μ_0 is position independent, it follows that the first term on the right hand side makes no contribution and we are left with

$$m\frac{\partial^2 \boldsymbol{\eta}}{\partial T^2} = -\nabla_{\mathbf{R}} \left[U_{ext}(\mathbf{R}) - U_{ext}(\mathbf{R} - \boldsymbol{\eta}) \right]. \tag{73}$$

For a harmonic trap potential described by

$$U_{ext}(\mathbf{R}) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2), \tag{74}$$

(73) reduces to (71), as claimed.

We next argue that the rigid in-phase oscillation described by (70) corresponds to the following distribution functions:

$$f_1(\mathbf{p}, \mathbf{R}, T) \equiv f_{10}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))$$

$$f_2(\mathbf{p}, \mathbf{R}, T) \equiv f_{20}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)), \tag{75}$$

where $f_{10}(\mathbf{p}, \mathbf{R})$ and $f_{20}(\mathbf{p}, \mathbf{R})$ satisfy the static equilibrium kinetic equations (69) and (67) and the static equilibrium densities satisfy (70). In this case, (38) for $f_1(\mathbf{p}, \mathbf{R}, T)$ reduces to

$$\left[\frac{\partial}{\partial T} - \nabla_{\mathbf{R}}\left[\tilde{\epsilon}_p(\mathbf{R}, T) + \mathbf{v}_s \cdot \mathbf{p}\right] \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}}\left[\tilde{\epsilon}_p(\mathbf{R}, T) + \mathbf{v}_s \cdot \mathbf{p}\right] \cdot \nabla_{\mathbf{R}}\right] f_{10}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta})$$
$$= -ig\left[m(\mathbf{R}, T)f_{20}(-\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}) - m^*(\mathbf{R}, T)f_{20}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta})\right] +$$
$$+ \frac{g}{2}\left[\nabla_{\mathbf{R}}m(\mathbf{R}, T) \cdot \nabla_{\mathbf{p}}f_2(-\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}) + \nabla_{\mathbf{R}}m(\mathbf{R}, T) \cdot \nabla_{\mathbf{p}}f_2(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta})\right]. \tag{76}$$

From the definition of $f_{i0}(\mathbf{p}, \mathbf{R})$, we see that $f_{i0}(-\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T)) = f_{i0}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}(T))$. We also note that for the in-phase oscillation under consideration, $m(\mathbf{R}, T) = m_0(\mathbf{R} - \boldsymbol{\eta}(T))$, where m_0 is real. Therefore the first term on the right side of (76) vanishes, leaving us with

$$\left[\frac{\partial}{\partial T} - \nabla_{\mathbf{R}}\left[\tilde{\epsilon}_p(\mathbf{R}, T) + \mathbf{v}_s \cdot \mathbf{p}\right] \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}}\left[\tilde{\epsilon}_p(\mathbf{R}, T) + \mathbf{v}_s \cdot \mathbf{p}\right] \cdot \nabla_{\mathbf{R}}\right] f_{10}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta})$$
$$= g\nabla_{\mathbf{R}}m_0(\mathbf{R} - \boldsymbol{\eta}) \cdot \nabla_{\mathbf{p}}f_{20}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}). \tag{77}$$

Using (39), (43) and (70), it is straightforward to see that $\tilde{\epsilon}_p(\mathbf{R}, T) = \tilde{\epsilon}_p(\mathbf{R} - \boldsymbol{\eta})$. Using this, (77) can be rewritten as

$$\left[\frac{\partial}{\partial T} - \nabla_{\mathbf{R}}\tilde{\epsilon}_p(\mathbf{R} - \boldsymbol{\eta}) \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}}\left[\tilde{\epsilon}_p(\mathbf{R} - \boldsymbol{\eta}) + \dot{\boldsymbol{\eta}} \cdot \mathbf{p}\right] \cdot \nabla_{\mathbf{R}}\right] f_{10}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta})$$
$$= g\nabla_{\mathbf{R}}m_0(\mathbf{R} - \boldsymbol{\eta}) \cdot \nabla_{\mathbf{p}}f_{20}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}). \tag{78}$$

If we introduce new variable $\mathbf{R}'(T) = \mathbf{R} - \boldsymbol{\eta}(T)$, and note that

$$\nabla_{\mathbf{R}} = \nabla_{\mathbf{R}'} \qquad \frac{\partial}{\partial T} = -\dot{\boldsymbol{\eta}} \cdot \nabla_{\mathbf{R}'}, \tag{79}$$

then (78) can be rewritten as

$$\left[-\dot{\boldsymbol{\eta}} \cdot \nabla_{\mathbf{R}'} - \nabla_{\mathbf{R}'}\tilde{\epsilon}_p(\mathbf{R}') \cdot \nabla_{\mathbf{p}} + \nabla_{\mathbf{p}}\tilde{\epsilon}_p(\mathbf{R}') \cdot \nabla_{\mathbf{R}'} + \dot{\boldsymbol{\eta}} \cdot \nabla_{\mathbf{R}'}\right] f_{10}(\mathbf{p}, \mathbf{R}')$$
$$= g\nabla_{\mathbf{R}'}m_0(\mathbf{R}') \cdot \nabla_{\mathbf{p}}f_{20}(\mathbf{p}, \mathbf{R}'). \tag{80}$$

Since the first and the last term on the left side of (80) cancel each other, we are left with an equation which is precisely the same as the static HFB equation (69) for the diagonal distribution function f_1.

Similarly, for the in-phase mode, Eq.(40) for the off-diagonal distribution function f_2 reduces to

$$\left[\frac{\partial}{\partial T} + 2i\tilde{\epsilon}_p(\mathbf{R}, T) + \dot{\boldsymbol{\eta}} \cdot \nabla_\mathbf{R} \right] f_{20}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}) = -igm_0(\mathbf{R} - \boldsymbol{\eta}) \left[2f_{10}(\mathbf{p}, \mathbf{R} - \boldsymbol{\eta}) + 1 \right]. \quad (81)$$

Using (79), (81) becomes

$$f_{20}(\mathbf{p}, \mathbf{R}') = -\frac{gm_0(\mathbf{R}')}{2\tilde{\epsilon}_p(\mathbf{R}')} \left[2f_{10}(\mathbf{p}, \mathbf{R}') + 1 \right]. \quad (82)$$

Again, this result is seen to be equivalent to the static HFB equation result given by (66) and (67).

In summary, we have explicitly verified that the ansatz given in (70) satisfies our coupled HFB kinetic equations for the two distribution function f_1 and f_2. The oscillating center-of-mass displacement $\boldsymbol{\eta}(T)$ satisfies the SHO equation of motion in (71). Thus, we have verified that our HFB coupled kinetic equations exhibit a solution corresponding to a rigid SHO oscillation of the static equilibrium density profiles ($n_c, \tilde{n}$ and $\tilde{m}$) in the direction of η_α, with the trap frequency ω_α. This expected solution is an important check on the correctness of our HFB equations of motion. We might note that we did not make use of the approximate semi-classical local equilibrium forms for f_1 and f_2 given in Section IV to prove that the Kohn mode solution exists.

VI. CONCLUSIONS

To summarize, in this paper, we have used the HFB self-energy approximation to derive the equation of motion for the condensate order parameter (given by (17)) and the kinetic equations for the diagonal and the off-diagonal distribution functions for the non-condensate atoms (given by (38) and (40)). We have only solved these coupled equations to exhibit the rigid in-phase Kohn mode (Section V). However, we emphasize that, more generally, these equations will lead to collective modes which satisfy various conservation laws [9], even

though they were generated from the HFB single-particle self-energies (which gives a single-particle spectrum with an energy gap in a uniform Bose gas). The essential physics is discussed in Sections 6 and 7 of Griffin's Varenna lectures [12]. Another way of saying this is that for a uniform Bose gas, the coupled equations of motion in this paper will lead to the same "conserving" density response function discussed by Cheung and Griffin [17].

Clearly the next step is to extend our present analysis by including the appropriate second-order self-energy contributions and hence, incorporate the effect of collisions into our kinetic equations. This generalization has been carried out for a uniform Bose-condensed gas by Kane and Kadanoff (KK) [2], work which has been recently extended to trapped Bose gases [18]. KK concentrated on using their kinetic equations to derive the two-fluid hydrodynamics equations. Kirkpatrick and Dorfman [11] also derived a kinetic equation for the distribution function describing excitations in a uniform Bose-condensed gas, using a different formalism than KK. We believe that the KK approach based on non-equilibrium Green's functions gives the most systematic approach to deriving generalized quantum Boltzamnn equations for trapped Bose gases.

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